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An application of algebraic R-local homotopy theory

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Abstract

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Let $R \subseteq \mathbb{Q}$ be a subring, let $r \geq 3$ and let m be an integer such that each prime p with $2p - 3 \leq m - r$ is invertible in R. Assume that the r-reduced R-local CW-complex C has R-dimension $\leq m$ and is a co-H-space. Then C is homotopy equivalent to a wedge of Moore spaces. If $H_*(C, R)$ is a free R-module, C is cogroup-like and $r \geq 4$, then C is co-H-equivalent to a suspension.

Let $r \geq 3$ be an integer. Let $R \subset \mathbb{Q}$ be a subring and $\bar{p}$ the smallest prime not invertible in R. Set $m = r + 2\bar{p} - 4$.

A CW-complex X is called *r-reduced*, if the $(r-1)$-skeleton of X is a point. An *R-local* sphere S^n_R, $n \geq 1$, is an n-reduced simple CW-complex with $H_i(S^n_R; \mathbb{Z}) = 0$ for $i \neq 0, n$ and $H_n(S^n_R; \mathbb{Z}) \cong R$. An r-reduced *R-local* CW-complex of *R-dimension* $\leq m$ is a cell complex built from a point by successively attaching reduced cones on R-local spheres S^n_R, $r - 1 \leq n \leq m - 1$.

The *R-localization* of a CW-complex X is denoted by X_R.

Let CW^m_r denote the class of R-local r-reduced CW-complexes of R-dimension $\leq m$.

Let Lie_R (resp. $\mathrm{d\,Lie}_R$) be the category of graded (resp. differential graded) Lie algebras over R.

In [1] it is shown that the homotopy category of CW^m_r is equivalent to the homotopy category of the category of free differential Lie algebras over R generated by elements x with $(r-1) \leq \mathrm{degree}(x) \leq m - 1$. In [9] this result is derived from tame homotopy theory. It is this version which will be used here.

For sake of simplicity we treat only the case $r \geq 3$; as in [1] corresponding results are true for $r = 2$.

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Theorem. *Let $C \in \mathrm{CW}_r^m$ be a co-H-space.*

(1) *Then C is homotopy equivalent to a wedge of Moore spaces.*

(2) *Let $H_*(C; R)$ be free as R-module and assume that C is a cogroup in the homotopy category. Then, for $r > 3$, C is co-H-equivalent to a suspension.*

In view of studies of mapping spaces (e.g. [5]) we mention the following immediate corollary:

Corollary. *Let Y be any pointed simply connected R-local CW-space and* $\mathrm{map}_*(C, Y)$ *the space of pointed maps $C \to Y$. If C is of finite type over R, then* $\mathrm{map}_*(C, Y)$ *is homotopy equivalent to a product of spaces $\Omega^{n_i}(Y)$, $F(p^{m_j}, j)$ where $F(p^{m_j}, j)$ is the homotopy fibre of the power map $\Omega^j(Y) \to \Omega^j(Y)$, $x \mapsto x^{p^{m_j}}$, p a prime which is non-invertible in R.*

Remark 1. The assertion of (1) should be well-known. Its archetype can be found in [2]. In the rational case part (2) is proved in [8] for $r \geq 3$ and (unpublished) for $r \geq 2$ by I. Berstein, J. Harper and J. Neisendorfer (see [7]). Therefore, in the R-local case, part (2) should also be true for $r \geq 3$.

Remark 2. The following dual version of the Theorem is true: Let X be an r-reduced, R-local H-space with $\pi_i(X) = 0$ for $i > m$. Then X is equivalent to a product of Eilenberg–Mac Lane spaces.

Let $\pi_*(X)$ be free as R-module and assume that X is a group in the homotopy category. Then X is H-equivalent to a loop space.

Proof of the Theorem. (i) Let S_k be the localization of $\mathbb{Z}$ at the set of primes p such that $2p - 3 > k$. Let R_*, $R_0 \subset R_1 \subset \cdots$, be the system of subrings of $\mathbb{Q}$ defined by $R_i = R.S_i$. We use the setting of tame homotopy theory with respect to R_* (over the ground ring R) as discussed in [9].

Let Σ (resp. Ω) denote reduced suspension (resp. space of loops).

Recall that a Moore space $M(A, n)$, $n \geq 1$, A an abelian group, is a simple space with reduced homology $\tilde{H}_i(M(A, n); \mathbb{Z}) = 0$ for $i \neq n$ and $\tilde{H}_n(M(A, n); \mathbb{Z}) \cong A$.

By [4] the space $\Sigma\Omega C$ is tamely equivalent to the wedge of Moore spaces $M(H_i(\Sigma\Omega C; R), i)$, $i > 0$. In fact, a model of $\Sigma\Omega C$ in $\mathrm{d\,Lie}_R$ is the free Lie algebra on the chain complex of ΩC. It follows, that up to degree m the Hurewicz map of $\Sigma\Omega C$ has a right inverse. (Recall that, since $\Sigma\Omega C$ is R-local, its mth Postnikov section coincides with the mth Postnikov section of a taming of $\Sigma\Omega C$.) By [6] C is a retract of $\Sigma\Omega C$; thus the Hurewicz map of C has a right inverse and hence $C \sim \bigvee_{i=r}^{m} M(H_i(C; R), i)$.

(ii) Let $V = s^{-1}\tilde{H}_*(C; R)$ where $(s^{-1}\tilde{H}_*(C; R))_q = \tilde{H}_{q+1}(C; R)$. Then by part (i) the homotopy type of C is represented in $\mathrm{d\,Lie}_R$ by the free (differentiable) Lie algebra $(\mathbb{L}(V), d = 0)$. The comultiplication of C then defines a morphism from $\mathbb{L}(V)$ to the coproduct $\mathbb{L}(V) \sqcup \mathbb{L}(V)$ which turns $\mathbb{L}(V)$ into a cogroup in Lie_R. Moreover, this cogroup structure is determined by the Berstein coalgebra $B(C)$ (see [8]; at this point the assumption "$H_*(C; R)$ is a free module" is essential). We note that $B(C)$ is cocommutative and $B(C)_q = 0$ for $0 < q < r - 1$, $q > m - 1$. Assume now that Y is a space in CW_{r-1}^{m-1} such that $H_*(Y, R) \cong B(C)$ as coalgebra. Then, $B(\Sigma Y) \cong B(C)$ by [3] and hence C and ΣY are co-H-equivalent. (Note that $H_*(\Omega\Sigma Y; R)$ is isomorphic to the universal enveloping algebra $U\mathbb{L}(V)$ by [1].)

To find such a space Y we apply the algebraization of tame homotopy theory via coalgebras studied in [11]. By this theory $B(C)$ determines a homotopy type Y' in tame homotopy theory with respect to R_* (and $r - 1 \geq 3$) such that $H_*(Y'; R)$ is isomorphic to $B(C)$ as coalgebra in degrees $\leq m - 1$. We then define Y as the $(m - 1)$th R-homology section of Y'. □

Proof of Remark 2. By [10] any tame H-space is equivalent to a product of Eilenberg–Mac Lane spaces; this implies the first part. To prove the second part first note that the H-space multiplication is determined by the Lie algebra structure of $\pi_*(X)$ (or the algebra structure of the universal enveloping algebra $U\pi_*(X)$). Hence it suffices to find a space Y with $\pi_*(\Omega Y) \cong \pi_*(X)$ (as Lie algebra). Choose a homotopy type Y' in tame homotopy theory with respect to R_* and $r + 1$ (instead of r) represented by $\pi_*(X)$ and define Y as the $(m + 1)$th Postnikov section of Y'. □

Proof of the Corollary. Being of finite type C is homotopy equivalent to a wedge $\bigvee M(A_i, n_i)$ of Moore spaces with monogenic R-modules A_i. Therefore, $\mathrm{map}_*(C, Y)$ is homotopy equivalent to the product of spaces $\mathrm{map}_*(M(A_i, n_i), Y)$. In case $A_i \cong R$, $\mathrm{map}_*(M(A_i, n_i), Y) \sim \Omega^{n_i}(Y)$. In case $A_i \cong \mathbb{Z}/q^i\mathbb{Z}$, q prime, $\mathrm{map}_*(M(A_i, n_i), Y) \sim F(q^i, n_i)$; this follows by applying $\mathrm{map}_*(-, Y)$ to the cofibre sequence

$$* \to S^{n_i} \xrightarrow{q^i} S^{n_i} \to M(\mathbb{Z}/q^i\mathbb{Z}, n_i) \to *$$

(where q^i denotes a map of degree q^i). For, the fibre $\mathrm{map}_*(M(\mathbb{Z}/q^i\mathbb{Z}, n_i), Y)$ of the fibration $\Omega^{n_i}(Y) \to \Omega^{n_i}(Y)$ induced by q^i is homotopy equivalent to $F(q^i, n_i)$. □

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